ON THE SOLUTION OF DYNAMIC PROBLEMS IN THE PLANE THEORY OF ELASTICITY FOR ANISOTROPIC MEDIA

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1. The equations of motion in terms of potentials. The equations of motion for the displacements in a plane anisotropic medium, in the absence of body forces, are [1]

$$a\frac{\partial^2 u}{\partial x^2}+d\frac{\partial^2 u}{\partial y^2}+c\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial^2 u}{\partial t^2}\qquad c\frac{\partial^2 u}{\partial x\partial y}+d\frac{\partial^2 v}{\partial x^2}+a\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 v}{\partial t^2}\qquad(1.1)$$

where u, v are the components of the displacement vector, a, c, d are elastic constants, the density of the medium having been taken equal to unity. We shall restrict attention to the case of three elastic constants, since the more general case can be treated similarly. Introducing the potentials of rotation free and equivoluminal displacements by means of the equations

$$u=\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y},\qquad v=\frac{\partial\varphi}{\partial y}-\frac{\partial\psi}{\partial x}\qquad(1.2)$$

we obtain the equations of motion in terms of potentials

$$\begin{aligned}\frac{\partial}{\partial x}\left[a\frac{\partial^2\varphi}{\partial x^2}+(d+c)\frac{\partial^2\varphi}{\partial y^2}-\frac{\partial^2\varphi}{\partial t^2}\right]+\frac{\partial}{\partial y}\left[(a-c)\frac{\partial^2\psi}{\partial x^2}+d\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial t^2}\right]=0\\ \frac{\partial}{\partial y}\left[(d+c)\frac{\partial^2\varphi}{\partial x^2}+a\frac{\partial^2\varphi}{\partial y^2}-\frac{\partial^2\varphi}{\partial t^2}\right]-\frac{\partial}{\partial x}\left[d\frac{\partial^2\psi}{\partial x^2}+(a-c)\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial t^2}\right]=0\end{aligned}\qquad(1.3)$$

A generalization of the method of complex solutions to the case of systems of homogeneous differential equations of the second order has already been given in [1]. These results apply immediately to the system (1.3) and furnish its solutions of the particular form

$$\varphi=\Phi(\Omega),\qquad\psi=\Psi(\Omega)\qquad(1.4)$$

where Ω is defined by

$$\delta \equiv l(\Omega)\, t + m(\Omega)\, x + n(\Omega)\, y + k(\Omega) = 0 \tag{1.5}$$

The following formulas are valid for the derivatives of the function ϕ (analogous formulas hold for the function ψ):

$$\begin{gathered}\frac{\partial^3 \varphi}{\partial x^{\alpha'} \partial y^{\beta'} \partial t^{\gamma'}} = -\frac{1}{\delta'}\frac{\partial}{\partial \Omega}\left[\frac{1}{\delta'}\frac{\partial}{\partial \Omega}\left(\frac{m^{\alpha'} n^{\beta'} l^{\gamma'}}{\delta'}\Phi'\right)\right] \qquad (\alpha' + \beta' + \gamma' = 3) \\ \delta' = l'(\Omega)\, t + m'(\Omega)\, x + n'(\Omega)\, y + k'(\Omega) \neq 0 \\ l'(\Omega) = \frac{dl}{d\Omega}, \ldots, \Phi'(\Omega) = \frac{d\Phi}{d\Omega}\end{gathered} \tag{1.6}$$

It is readily seen that the system (1.3) is satisfied provided that l, m, n are such that

$$\begin{gathered} m\,[am^2 + (d + c)\, n^2 - l^2]\, \Phi' + n\,[(a - c)\, m^2 + dn^2 - l^2]\, \Psi' = 0 \\ n\,[(d + c)\, m^2 + an^2 - l^2]\, \Phi' - m\,[dm^2 + (a - c)\, n^2 - l^2]\, \Psi' = 0 \end{gathered} \tag{1.7}$$

which implies that the following relation must hold:

$$\begin{vmatrix} m\,[am^2 + (d + c)\, n^2 - l^2] & n\,[(a - c)\, m^2 + dn^2 - l^2] \\ n\,[(d + c)\, m^2 + an^2 - l^2] & -m\,[dm^2 + (a - c)\, n^2 - l^2] \end{vmatrix} = 0 \tag{1.8}$$

and a similar relation between Φ' and Ψ'. Putting $l \equiv 1$, $m = -\theta$, $n = \lambda$, we may rewrite (1.5) in the form

$$\delta_j \equiv t - \theta_j x + \lambda_j(\theta_j)\, y + k_j(\theta_j) = 0 \tag{1.9}$$

where λ_j are the roots of Equation (1.8), which may be written

$$\lambda^4 - \frac{a + d - L\theta^2}{ad}\lambda^2 + \left(\frac{1}{a} - \theta^2\right)\left(\frac{1}{d} - \theta^2\right) = 0 \qquad (L = a^2 + d^2 - c^2) \tag{1.10}$$

Obviously the λ_i are the branches of an algebraic function λ which is single-valued on a Riemann surface which consists of two planes θ_1 and θ_2, cut, respectively, along the intervals $(-1/\sqrt{a},\ 1/\sqrt{a})$, $(-1/\sqrt{d},\ 1/\sqrt{d})$. The planes are attached to each other along a cut that joins the branch points θ_k°, which are the roots of the equation

$$\left(\frac{a + d - L\theta^2}{2ad}\right)^2 - \left(\frac{1}{a} - \theta^2\right)\left(\frac{1}{d} - \theta^2\right) = 0 \tag{1.11}$$

These roots are not real but complex conjugates, provided that $c < a - d$. This inequality holds for all anisotropic bodies which are considered in [2]; as may be seen from the following table [2], where unit stress is taken to be 10^6 g/cm^2.

Medium	a	d	c	$a-d$
Pyrites (cubic)	3680	1075	592	2505
Fluor Spar	1670	345	797	1325
Rock-salt	477	129	261	348
Potassium chloride	375	65.5	263.5	309.5

In order to construct solutions we shall employ the first of the relations (1.7). By introducing the functions Φ_j and Ψ_j, corresponding to the the root λ_j, we obtain

$$\Phi_j'(\theta_j) = \lambda_j P_j(\theta_j)\,\omega_j(\theta_j), \qquad \Psi_j'(\theta_j) = \theta_j Q_j(\theta_j)\,\omega_j(\theta_j) \tag{1.12}$$

where ω_j is a branch of an arbitrary algebraic function ω which is single-valued on the Riemann surface mentioned above, and

$$P_j(\theta_j) = (a-c)\,\theta_j^2 + d\lambda_j^2 - 1, \qquad Q_j(\theta_j) = a\theta_j^2 + (d+c)\,\lambda_j^2 - 1 \tag{1.13}$$

The general real-valued solution (of the form (1.4)) of the system (1.3) is given by

$$\varphi(x,y,t) = \sum_{j=1}^{2} \operatorname{Re} \int^{\Theta_j} \lambda_j P_j(\xi)\,\omega_j(\xi)\,d\xi, \qquad \psi(x,y,t) = \sum_{j=1}^{2} \operatorname{Re} \int^{\Theta_j} \xi Q_j(\xi)\,\omega_j(\xi)\,d\xi \tag{1.14}$$

In order to obtain the homogeneous solutions of zero order, one has to set $k_j \equiv 0$ in (1.9); this yields

$$\delta_j = 1 - \theta_j \xi + \lambda_j(\theta_j)\,\eta = 0 \qquad \left(\xi = \frac{x}{t},\ \ \eta = \frac{y}{t}\right) \tag{1.15}$$

which furnishes the correspondence between the above-mentioned Riemann surface and the domain in the $\xi\eta$-plane, where the functions $\theta_1(\xi, \eta)$ and $\theta_2(\xi, \eta)$ are defined. This is a double-sheeted domain, consisting of two separate domains, corresponding to the planes θ_1 and θ_2, attached along the cut which joins the branch points $(\xi_k^\circ, \eta_k^\circ)$. These two points are the images, in the $\xi\eta$-plane, of the branch points θ_k°. The boundaries of these domains are obtained as the envelopes of the straight lines (1.15) for real θ_j and λ_j. Solving Equation (1.10), we obtain

$$\lambda_j = \left[\frac{a+d-L\theta^2}{2ad} + (-1)^j \sqrt{\left(\frac{a+d-L\theta^2}{2ad}\right)^2 - \left(\frac{1}{a} - \theta^2\right)\left(\frac{1}{d} - \theta^2\right)}\right]^{1/2}$$

where the radical sign inside the square brackets refers to that branch which is positive for real θ, and the "outer" square root refers to the branch which is positive on the upper banks of the cuts $(-1/\sqrt{a},\ 1/\sqrt{a})$, $(-1/\sqrt{d},\ 1/\sqrt{d})$. For $c < a - d$, outside these cuts on the real axis, λ_1 and λ_2 take on only purely imaginary values. Thus the points of the Riemann surface which lie on the mentioned cuts correspond, in the $\xi\eta$-plane, to points of curves on the cone of rays; and the point at infinity corresponds to the origin of coordinates. The two-sheeted domain in the $\xi\eta$-plane is the simultaneous domain of definition of the functions θ_1 and θ_2. In the xyt-space it determines the interior of a characteristic cone of the system (1.3), with vertex at the point $x = y = t = 0$.

2. Lamb's problem. Suppose first that on the boundary of an anisotropic half-space $y \leqslant 0$ there act the distributed tractions:

$$\sigma_y = -N(x, t), \quad \tau_{xy} = -T(x, t) \qquad \text{for } y = 0 \tag{2.1}$$

which differ from zero on the rectangle $0 < t < t_0$, $-l_1 < x < l_1$, and suppose that they have a finite impulse; and introduce the new tractions

$$N_\varepsilon(x, t) = \frac{1}{\varepsilon^2} N\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right), \qquad T_\varepsilon(x, t) = \frac{1}{\varepsilon^2} T\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \tag{2.2}$$

which differ from zero on the internal $0 < t < \epsilon t_0$, $-\epsilon l_1 < x < \epsilon l_1$, and let $\phi_\epsilon(x, y, t)$, $\psi_\epsilon(x, y, t)$ be the corresponding potentials. It is easy to show that in the limit, as $\epsilon \to 0$, we obtain homogeneous functions of the first order, ϕ and ψ, which correspond to the action of an instantaneous impulse. Thus in order to solve the problem it is necessary to obtain solutions of the system (1.3), with zero stress components σ_y and τ_{xy} on the boundary of the domain for $t > 0$, that is

$$\begin{gathered}(\delta_1 - 1)\frac{\partial^2\varphi}{\partial x^2} + \gamma_1\frac{\partial^2\varphi}{\partial y^2} - (1 + \gamma_1 - \delta_1)\frac{\partial^2\psi}{\partial x\,\partial y} = 0 \\ 2\frac{\partial^2\varphi}{\partial x\,\partial y} + \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2} = 0 \qquad \left(\delta_1 = \frac{c}{d},\ \gamma_1 = \frac{a}{d}\right)\end{gathered} \tag{2.3}$$

The solution will be sought in the form (1.14). The boundary conditions will then be automatically satisfied, provided that the analytic functions and Φ_j, Ψ_j, analytic in the upper half-plane, fulfill the relations

$$\begin{gathered}\sum_{j=1}^{2} \operatorname{Re}\{[(c-d)\,\theta^2 + a\lambda_j^2]\,\Phi_j' + (a+d-c)\,\theta\lambda_j\Psi_j'\} = 0 \\ \sum \operatorname{Re}\{-2\theta\lambda_j\Phi_j' + (\lambda_j^2 - \theta^2)\,\Psi_j'\} = 0\end{gathered} \tag{2.4}$$

Using (1.12), we obtain

$$\operatorname{Re}\,[\lambda_1 S_1\omega_1 + \lambda_2 S_2\omega_2] = 0, \qquad \operatorname{Re}\,\theta\,[M_1\omega_1 + M_2\omega_2] = 0 \tag{2.5}$$

from which it follows that

$$\lambda_1 S_1 \omega_1 + \lambda_2 S_2 \omega_2 = i\alpha, \qquad \theta\,[M_1\omega_1 + M_2\omega_2] = i\beta \tag{2.6}$$

where

$$\begin{aligned} S_j &= [(\delta_1 - 1)\,\theta^2 + \gamma_1\lambda_j^2]\,P_j + (1 + \gamma_1 - \delta_1)\,Q_j \\ M_j &= -\,2\lambda_j^2 P_j + (\lambda_j^2 - \theta^2)\,Q_j \end{aligned} \qquad (j = 1, 2) \tag{2.7}$$

Using the ω_j, as determined by (2.6), in Equation (1.12), we obtain

$$\Phi_1' + \Phi_2' = i\sum_{j=1}^{2} \frac{\lambda_j P_j M_{3-j}\,\alpha + \lambda_j\lambda_{3-j} P_{3-j} S_j\,\beta}{(\lambda_j - \lambda_{3-j})\,\Delta(\theta_j)}$$

$$\Psi_1' + \Psi_2 = i\sum_{j=1}^{2} \frac{\theta_j Q_j M_{3-j}\,\alpha + \lambda_j S_j Q_{3-j}\,\beta}{(\lambda_j - \lambda_{3-j})\,\Delta(\theta_j)} \qquad \Delta(\theta_j) = \theta_j \sum_{k=1}^{2} \frac{\lambda_k S_k M_{3-k}}{(\lambda_k - \lambda_{3-k})} \tag{2.8}$$

Since the immediate determination of the constants α and β is not an easy matter, we shall follow an indirect approach. We shall construct the solution of the same problem by means of a successive application of the Fourier and Laplace transformations, as is done in [3], and later compare this solution with (2.8); in this way we arrive at the result

$$\begin{aligned} \varphi &= -\frac{N_\rightarrow}{\pi d_1}\int_0^\infty R_1^{\circ\circ}(y, t, k)\frac{\cos kx}{k}\,dk + \frac{T_\rightarrow}{\pi d_1}\int_0^\infty R_2^{\circ\circ}(y, t, k)\frac{\sin kx}{k}\,dk \\ \psi &= \frac{N_\rightarrow}{\pi d_1}\int_0^\infty S_1^{\circ\circ}(y, t, k)\frac{\sin kx}{k}\,dk + \frac{T_\rightarrow}{\pi d_1}\int_0^\infty S_2^{\circ\circ}(y, t, k)\frac{\cos kx}{k}\,dk \end{aligned} \tag{2.9}$$

Here $d_1 = \rho d$, where ρ is the density of the medium, and $N_\rightarrow$, $T_\rightarrow$ are the normal and tangential components of the impulse

$$R_1^{\circ\circ} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\sum_{j=1}^{2}\frac{\lambda_j^{\circ\circ} P_j^{\circ\circ} M_{3-j}^{\circ\circ}}{\lambda_j^{\circ\circ} - \lambda_{3-j}^{\circ\circ}}\,e^{\vartheta}\,\frac{d\zeta}{\Delta^{\circ\circ}(\zeta)}$$

$$R_2^{\circ\circ} = \frac{1}{2\lambda i}\int_{\sigma-i\infty}^{\sigma+i\infty}\sum_{j=1}^{2}\frac{\lambda_j^{\circ\circ} P_j^{\circ\circ}\lambda_{3-j}^{\circ\circ} S_{3-j}^{\circ\circ\prime}\, l^{\vartheta}}{\lambda_j^{\circ\circ} - \lambda_{3-j}^{\circ\circ}}\,\frac{d\zeta}{\Delta^{\circ\circ}(\zeta)} \tag{2.10}$$

$$S_1^{\circ\circ} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\sum_{j=1}^{2}\frac{Q_j^{\circ\circ} M_{3-j}^{\circ\circ} e^{\upsilon}}{\lambda_j^{\circ\circ} - \lambda_{3-j}^{\circ\circ}}\,\frac{d\zeta}{\Delta^{\circ\circ}(\zeta)}, \qquad S_2^{\circ\circ} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\sum_{j=1}^{2}\frac{Q_j^{\circ\circ}\lambda_{3-i}^{\circ\circ} S_{3-i}^{\circ\circ}\, l}{\lambda_j^{\circ} - \lambda_{3-j}^{\circ\circ}}\,\frac{d\zeta}{\Delta^{\circ\circ}(\zeta)}$$

$$\vartheta = \vartheta(\zeta) = (\zeta t - \lambda_j^{\circ\circ} y^{\circ})\,k$$

The functions $\lambda_j^{\circ\circ}$, $M_j^{\circ\circ}$, $S_j^{\circ\circ}$, $P_j^{\circ\circ}$, $Q_j^{\circ\circ}$ are obtained from the λ_j, M_j, S_j, P_j, Q_j which were introduced earlier, by replacing θ by i/ζ. In order to determine α and β, one must compute, for example, $\partial\phi/\partial x$. From the solutions (2.9) we obtain

$$\frac{\partial\varphi}{\partial x}=\frac{N_{\rightarrow}}{\pi d_1}\int\limits_0^{\infty}\int\limits_{l_2}\sum_{j=1}^{2}\frac{\lambda_j^{\circ\circ}P_j^{\circ\circ}M_{3-j}^{\circ\circ}e^{\theta^{\circ\circ}}}{(\lambda_j^{\circ\circ}-\lambda_{3-j}^{\circ\circ})(ix-\lambda_j^{\circ\circ}y+\zeta t)}\frac{d\zeta\,dk}{i\Delta^{\circ\circ}(\zeta)}+$$
$$+T_{\rightarrow}\int\limits_0^{\infty}\int\limits_{l_2}\sum_{j=1}^{2}\frac{\lambda_i^{\circ\circ}\lambda_{3-i}^{\circ\circ}P_j^{\circ\circ}S_j^{\circ\circ\prime}e^{\theta^{\circ\circ}}d\zeta dk}{(\lambda_j^{\circ\circ}-\lambda_{3-j}^{\circ\circ})(ix-\lambda_j^{\circ\circ}y+\zeta t)\,\Delta^{\circ\circ}(\zeta)} \tag{2.11}$$
$$\theta^{\circ\circ}=\theta^{\circ\circ}(\zeta)=(ix-\zeta t-\lambda_j^{\circ\circ}y)\,k$$

Interchanging the order of integrations, choosing the contour l_2 in such a manner that it encloses only the singularities corresponding to the roots of the equation $ix-\lambda_j^{\circ\circ}y+\zeta t=0$; employing Jordan's lemma and the theorem of residues of Cauchy, we obtain, in terms of the variable θ:

$$\frac{\partial\varphi}{\partial x}=\sum_{j=1}^{2}\left\{-\frac{N_{\rightarrow}}{\pi d_1}\,\mathrm{Re}\frac{\lambda_jP_{3-j}M_{3-j}}{(\lambda_j-\lambda_{3-j})\,\delta_j'}\frac{i}{\Delta(\theta_j)}-\frac{T_{\rightarrow}}{\pi d_1}\,\mathrm{Re}\frac{\lambda_j\lambda_{3-j}P_jS_{3-j}}{(\lambda_j-\lambda_{3-j})\,\delta_j'}\frac{i}{\Delta(\theta_j)}\right\} \tag{2.12}$$
$$\delta_j'=-x+\lambda_j'(\theta_i)\,y$$

Upon comparison of this equation with the one resulting from the fundamental equation (2.8), it follows that

$$\alpha=-\frac{N_{\rightarrow}}{\pi d_1},\qquad \beta=-\frac{T_{\rightarrow}}{\pi d_1} \tag{2.13}$$

which completes the solution of the problem posed at the outset. The determinant $\Delta(\theta)$ is given by

$$\Delta(\theta)=-c\theta\,(\theta^2+\lambda_1^2)\,(\theta^2+\lambda_2^2)\sqrt{a^{-1}-\theta^2}\,\mathrm{R}(\theta)$$
$$R(\theta)=\{[a^2-(c-d)^2]\,\theta^2-a\}\sqrt{d^{-1}-\theta^2}-a\sqrt{a^{-1}-\theta^2} \tag{2.14}$$

Rayleigh's function $R(\theta)$ for the anisotropic media was studied in [1]. It has two real symmetric roots, which correspond to the speed of propagation of Rayleigh waves on the surface of the given medium. The qualitative picture of the motion in an anisotropic medium is analogous to the motion in an isotropic medium. A disturbance which originates at the origin of coordinates at the time $t=0$ is propagated throughout the entire half-space, dying out gradually at all interior points. With the passage of time almost all the energy of the disturbance is concentrated in the neighborhood of the surface of the medium and behaves, at sufficiently large distances from the center of the disturbance, as a Rayleigh surface wave. For $c<a-d$ we obtain from (2.8) the known solution of Lamb's problem for an isotropic medium.

3. Lamb's problem with mixed boundary conditions. The boundary conditions will be taken as in [4]:

$$\tau_{xy} = 0 \quad \text{for } y = 0, \; -\infty < x < \infty$$
$$\sigma_y = 0 \quad \text{for } x > 0, \; v = 0 \quad \text{for } x < 0 \tag{3.1}$$

The condition $\tau_{xy} = 0$ at boundary points yields

$$\theta \, (M_1\omega_1 + M_2\omega_2) = i\beta \tag{3.2}$$

Putting

$$\lambda_1 S_1\omega_1 + \lambda_2 S_2\omega_2 = A\,(\theta), \; \operatorname{Re} A\,(\theta) = 0 \qquad (\theta > 0) \tag{3.3}$$

there occurs the sought function $A(\theta)$, which will be supposed to be bounded at infinity. The third boundary condition gives

$$\sum_{j=1}^{2} (\lambda_j^2 P_j + \theta^2 Q_j)\,\omega_j = B\,(\theta), \qquad \operatorname{Re} B\,(\theta) = 0 \qquad (\theta < 0) \tag{3.4}$$

while (3.1) and (3.3) together give

$$\omega_j\,(\theta) = -\frac{\lambda_{3-j} S_{3-j}\, i\beta - \theta M_{3-j}\, A\,(\theta)}{(\lambda_j - \lambda_{3-j})\,\Delta\,(\theta_j)} \tag{3.5}$$

Substituting into (3.4), we then obtain

$$T_1{}^\circ A\,(\theta) - T_2{}^\circ i\beta = B\,(\theta) \tag{3.6}$$

where

$$T_1{}^\circ\,(\theta) = \sum_{j=1}^{2} \frac{\lambda_j P_j + \theta^2 Q_j}{\lambda_j - \lambda_{3-j}} \frac{\theta M_{3-j}}{\Delta\,(\theta)}, \quad T_2{}^\circ\,(\theta) = \sum_{j=1}^{2} \frac{\lambda_j^2 P_j + \theta^2 Q_j}{\lambda_j - \lambda_{3-j}} \frac{\lambda_{3-j} S_{3-j}}{\Delta\,(\theta)} \tag{3.7}$$

In the sequel it will be convenient to replace the constants a, d, c, respectively, by a^{-2}, d^{-2}, c^{-2}. Thus, the sought function $A(\theta)$, which is to be analytic in the upper half-plane, will satisfy on the real axis the conditions

$$\operatorname{Re} A\,(\theta) = 0 \quad (\theta > -a), \quad \operatorname{Re}\,[T_1{}^\circ A\,(\theta) - T_2{}^\circ i\beta] = 0 \quad (\theta < -a) \tag{3.8}$$

Since the functions P_j, Q_j, S_j, M are real for real values of the variable θ, in view of the value of the determinant $\Delta(\theta)$ and of the choice of the branches λ_j and of the roots $\sqrt{(a^2 - \theta^2)}$ and $\sqrt{(d^2 - \theta^2)}$ (they are supposed to be positive on the upper banks of the cuts $(-a, a)$, $(-d, d)$), we have that the function T_2 must be real and the function T_1 must be purely imaginary when $\theta < -d$. Consequently for $\theta < -d$ we must have that: $\operatorname{Im} A(\theta) = 0$, that is to say, the function $A(\theta)$ may be continued analytically across this segment of the axis. Denoting by f its real part on the segment $(-d, -a)$, we get

$$A(\theta) = \frac{\sqrt{d+\theta}}{2\pi i} \int_{-d}^{-a} \frac{f(\xi)\, d\xi}{\sqrt{d+\xi}\,(\xi-\theta)} = \sqrt{d+\theta}\,\chi(\theta) \tag{3.9}$$

By the radical $\sqrt{(d+\theta)}$ is to be understood here that branch which is positive on the upper bank of the cut $\theta > -d$. For the function $\chi(\theta)$, on the same upper bank, we obtain

$$\frac{\chi^+}{(\lambda_1-\lambda_2)\,\Delta(\theta)} + \frac{\overline{\chi^+}}{(\bar\lambda_1-\lambda_2)\,\bar\Delta(\theta)} = 2\,\frac{\operatorname{Re} T_2{}^\circ i\beta}{\sqrt{d+\theta}\,Q_0(\theta)}$$
$$Q_0 = \sum_{j=1}^{2} (-1)^{3-j}(\lambda_j{}^2 P_j + \theta^2 Q_j)\, M_{3-j} \tag{3.10}$$

Observing the value of $\Delta(\theta)$, and the fact that $\overline{\chi^+} = -\chi^-$, where χ^- is the value attained by $\chi(\theta)$ when approaching the same segment of the real axis from below, we deduce that

$$\chi^+ = G\chi^- + g \tag{3.11}$$
$$G(\theta) = -G_1(\theta) = -\frac{(\lambda_1-\lambda_2)\,R(\theta)}{(\bar\lambda_1-\lambda_2)\,\overline{R(\theta)}}, \qquad g(\theta) = \frac{2 Rei T_2{}^\circ (\lambda_1-\lambda_2)\,\Delta(\theta)}{\sqrt{d+\theta}\,Q_0(\theta)}\,\beta$$

Thus we are led to a well-known boundary-value problem, whose solution, satisfying all the stated conditions, can be put in the form

$$\chi(\theta) = \beta\,\frac{X_0(\theta)}{2\pi i} \int_{-d}^{-a} \frac{g_1(\xi)\,(\lambda_1-\lambda_2)\,\Delta(\xi)}{X_0{}^+(\xi)\,(\xi-0)}\, d\xi + X_0(\theta)\, i\beta_1 \tag{3.12}$$

$$g_1(\theta) = \frac{2 Rei T_2{}^\circ}{\sqrt{d+0}\,Q_0(\theta)}, \qquad X_0(\theta) = \frac{1}{\sqrt{d+\theta}\,\sqrt{a+\theta}} \exp \frac{1}{2\pi i} \int_{-d}^{-a} \frac{\ln G_1\, d\xi}{\xi-\theta}$$

In view of (2.8) we have

$$\Phi_1' + \Phi_2' = \sum_{j=1}^{2} \frac{(\theta_j M_{3-j} A(\theta) - \lambda_{3-j} S_{3-j} i\beta)\,\lambda_j P_j}{(\lambda_j - \lambda_{3-j})\,\Delta(\theta_j)}$$
$$\Psi_1' + \Psi_2' = \sum_{j=1}^{2} \frac{(\theta_j M_{3-j} A(\theta) - \lambda_{3-j} S_{3-j} i\beta)\,\theta_j Q_j}{(\lambda_j - \lambda_{3-j})\,\Delta(\theta_j)} \tag{3.13}$$

The constant β_1 may be obtained from the condition that the solution must be bounded for $\theta = -c_0$, where $c_0{}^{-1}$ is the speed of the Rayleigh waves. In view of (3.13) this means that

$$\sum_{j=1}^{2} \frac{\theta M_{3-j}\lambda_j P_j}{\lambda_j - \lambda_{3-j}}\, A(\theta) - \sum_{j=1}^{2} \frac{\lambda_j \lambda_{3-j} P_j S_{3-j}}{\lambda_j - \lambda_{3-j}}\, i\beta = 0 \qquad \text{for } \theta = -c_0 \tag{3.14}$$

which in turn implies that, since $\Delta(-c_0) = 0$, that the numerator of the expression for $\Psi_1' + \Psi_2'$ is also zero, thus enabling us to evaluate the constant β_1 by means of the constant β, and to write that

$$A(\theta) = i\beta A_0(\theta) \tag{3.15}$$

Substituting this into (3.13), we obtain

$$\Phi_1' + \Phi_2' = i\beta \sum_{j=1}^{2} \frac{(\theta_j M_{3-j} A_0(\theta) - \lambda_{3-j} S_{3-j}) \lambda_j P_j}{(\lambda_j - \lambda_{3-j}) \Delta(\theta)} \tag{3.16}$$

$$\Psi_1' + \Psi_2' = i\beta \sum_{j=1}^{2} \frac{(\theta_j M_{3-j} A_0(\theta) - \lambda_{3-j} S_{3-j}) \theta_j Q_j}{(\lambda_j - \lambda_{3-j}) \Delta(\theta_j)} \tag{3.17}$$

Since for large θ the term containing $A_0(\theta)$ in (3.17) tends to zero, and Expression (3.17) tends to the solution which corresponds to the action of a purely tangential component of the impulse, we must have that $\beta = -T_{\rightarrow}/\pi d_1$; and thus the problem has been entirely solved. It may be readily verified that when $c^{-2} = a^{-2} - d^{-2}$, we are led back to the results obtained in the isotropic case in [4].

4. Reflection of plane waves from rectilinear boundaries. The consideration of the reflection of a plane wave from a rectilinear boundary leads to a homogeneous Hilbert problem. The evolution of the wave is given in the form

$$\begin{gathered} \varphi_1^\circ(\Omega_1^\circ) + \varphi_2^\circ(\Omega_2^\circ) = \sum_{j=1}^{2} \lambda_j^\circ P_j^\circ \omega_j(\Omega_j^\circ) \\ \Omega_j^\circ = t - \theta_0 x - \lambda_j(\theta_0) y \\ \psi_1^\circ(\Omega_1^\circ) + \psi_2^\circ(\Omega_2^\circ) = \sum_{j=1}^{2} \theta_0 Q_j^\circ \omega_j(\Omega_j^\circ) \end{gathered} \tag{4.1}$$

where the ω_j are branches of functions which are single-valued on the above-mentioned Riemann surface. The boundary conditions are mixed (see Fig. 1; notice that in Figs. 1 and 3 the points ξ_k° are the image points, in the plane xy at the instant t, of the branch points θ_k°). The reflected waves, corresponding to various boundary conditions, may be easily constructed for $x > 0$ and $x < 0$. The solution is obtained in the form

$$\varphi_1 + \varphi_2 = \sum_{j=1}^{2} (\varphi_j^\circ + \varphi_{-j}^\circ + \varphi^\circ_{j-3}{}^{\circ\circ}), \quad \psi_1 + \psi_2 = \sum_{j=1}^{2} (\psi_j^\circ + \psi^\circ_{-j} + \psi_{j-3}{}^{\circ\circ}) \tag{4.2}$$

where

$$\varphi_j = -\lambda_j^\circ P_j^\circ \omega_j(\Omega_j^\circ) + \lambda_j^\circ P_j^\circ C_j \omega_j(\Omega_{-j}^\circ) + \lambda_{3-j}^\circ P_{3-j}^\circ D_{3-j} \omega_j(\Omega_{j-3}^\circ)$$

$$\psi_j = \theta_0 Q_j^\circ \omega_j(\Omega_j^\circ) + \theta_0 Q_j^\circ C_j \omega_j(\Omega_{-j}^\circ) + \theta_0 Q_{3-j} D_{3-j} \omega_j(\Omega_{j-3}^\circ) \qquad (4.3)$$

$$\lambda_j^\circ = \lambda_j(\theta_0), \quad P_j^\circ = P_j(\theta_0), \quad Q_j^\circ = Q_j(\theta_0), \; \Omega^\circ_{-j} = t - \theta_0 x + \lambda^\circ_j, \ldots$$

and the values of the constants C_j and D_j depend on the boundary conditions. Thus for stress-free boundaries we have

$$C_j = \frac{\lambda_j^\circ S_j^\circ M_{3-j}^\circ + \lambda_{3-j}^\circ S_{3-j}^\circ M_j^\circ}{(\lambda_j^\circ - \lambda_{3-j}^\circ)\,\Delta^\circ}, \quad D_{3-j} = -\frac{2\lambda_j^\circ S_j^\circ M_j^\circ}{(\lambda_j^\circ - \lambda_{3-j}^\circ)\,\Delta^\circ}$$

$$\Delta^\circ = \Delta(\theta_0) \qquad (4.4)$$

and for the boundary conditions applicable for $x < 0$ we have

$$C_1' = D_1' = -1, \qquad C_2' = D_2' = 0 \qquad (4.5)$$

In the sequel we shall take for ω_j a step function $\omega_0(\xi)$, which equals zero for $\xi > 0$ and equals unity for $\xi < 0$. For this function the corresponding irrotational disturbance, in the domain CFD, corresponding

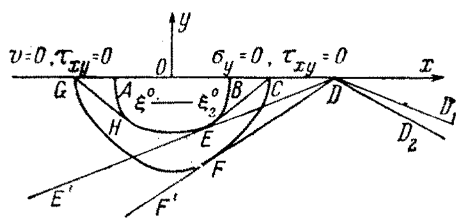

Fig. 1.

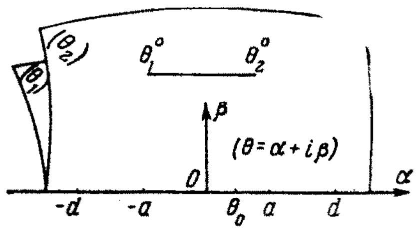

Fig. 2.

to the root λ_1, is just $\phi^\circ + \phi_{-1}^\circ + \phi_{-1}^{\circ\circ}$; while in the domain CED the disturbance corresponding to the root λ_2 equals $\phi^\circ + \phi_{-2}^\circ + \phi_{-2}^{\circ\circ}$. Similarly, the irrotational disturbances in the domains GFF' and $GHEE'$ have intensities $\phi_1 + \phi_{-1}^{\circ\prime} + \phi_{-1}^{\prime\circ\circ}$ and $\phi_2^\circ + \phi_{-2}^{\circ\prime} + \phi_{-2}^{\prime\circ\circ}$, respectively. In these domains one may also readily determine the intensity of the corresponding equivoluminal disturbances $\Psi_j^\circ + \Psi_{-j}^\circ + \Psi_{-j}^{\circ\circ}$ and $\Psi_j + \Psi_{-j}^{\circ\prime} + \Psi_{-j}^{\prime\circ\circ}$. The functions $\Phi_j(\theta_j)$ and $\Psi_j(\theta_j)$, which describe the disturbance in the domain $OGFCBO$ are defined in the upper half-planes of the Riemann surface. Since the arcs CFG and BEA are the envelopes of the straight lines $t - \theta_j x + \lambda_j y = 0$ for real values of θ_j and λ_j, the points E and F in the xy-plane must correspond to a single point θ_0, lying on the segment $(-a, a)$ (see Fig. 2). At this point the single-valued and piecewise constant (on the mentioned interval) functions $\Phi_1(\theta_1) + \Phi_2(\theta_2)$ and $\Psi_1(\theta_1) + \Psi_2(\theta_2)$ possess a finite discontinuity. Representing these functions by means of integrals of Cauchy type and differentiating, we easily obtain in the neighborhood of θ_0

$$\Phi_1'(\theta)+\Phi_2'(\theta)\approx\frac{\alpha_0}{\pi i(\theta-\theta_0)},\qquad \alpha_0=-2\frac{\lambda_1^\circ S_1^\circ+\lambda_2^\circ S_2^\circ}{\Delta^\circ}\sum_{j=1}^{2}\frac{\lambda_j^\circ P_j^\circ M_{3-j}}{\lambda_j^\circ-\lambda_{3-j}^\circ} \tag{4.6}$$

$$\Psi_1'(\theta)+\Psi_2'(\theta)\approx\frac{\beta_0}{\pi i(\theta-\theta_0)},\qquad \beta_0=-2\theta_0\frac{\lambda_1^\circ S_1^\circ+\lambda_2^\circ S_2^\circ}{\Delta^\circ}\sum_{j=1}^{2}\frac{Q_j^\circ M_{3-j}}{\lambda_j^\circ-\lambda_{3-j}^\circ}$$

The fact that the shear stress is zero on the boundary of the half-plane gives

$$\operatorname{Re}(M_1\omega_1+M_2\omega_2)=0 \tag{4.7}$$

The fact that there is zero normal stress for $x > 0$ gives

$$\operatorname{Re}(\lambda_1 S_1\omega_1+\lambda_2 S_2\omega_2)=0\qquad(\theta>0) \tag{4.8}$$

The absence of vertical displacement for $x < 0$ gives

$$\operatorname{Re}(T_1\omega_1+T_2\omega_2)=0,\qquad T_j=\lambda_j^2P_j+\theta^2Q_j\qquad(\theta<0) \tag{4.9}$$

Since there is no source of vibrations at the origin, it follows that

$$M_1\omega_1+M_2\omega_2=0$$

and putting $\lambda_1S_1\omega_1+\lambda_2S_2\omega_2=A(\theta)$, we have

$$\operatorname{Re}A(\theta)=0,\ \theta>0$$

Expressing ω_j by means of $A(\theta)$ and substituting in (4.13), we obtain

$$\operatorname{Re}\frac{T_1M_2-T_2M_1}{\lambda_1-\lambda_2}\frac{A(\theta)}{\Delta(\theta)}=0\qquad(\theta<0) \tag{4.10}$$

It is readily seen that the imaginary part of $A(\theta)$ is zero for $\theta<-d$; hence according to (3.9) we obtain $A=\sqrt{(d+\theta}A_1)$, where

$$\operatorname{Re}\frac{T_1M_2-T_2M_1}{\lambda_1-\lambda_2}\frac{A_1(\theta)}{\Delta(\theta)}=0\qquad(-d<\theta<-a) \tag{4.11}$$

The function $A_1(\theta)$ has a first-order pole at the point $\theta=\theta_0$, while the resulting solution, as before, must be bounded for $\theta=-c_0$. Introducing the new function $A_2=(\theta-\theta_0)A_1/(\theta-c_0)$, we obtain on the upper bank of the segment $(-d,\ -a)$

$$\operatorname{Re}\frac{T_1M_2-T_2M_1}{\lambda_1-\lambda_1}\frac{A_2^+(\theta)}{\sqrt{a^2-\theta^2}R(\theta)}=0\qquad(-d<\theta<-a) \tag{4.12}$$

where A_2^+ is the limiting value of the function A_2 on the segment $(-d,$ $-a)$ when this segment is approached from above. Denoting by A_2^- its boundary value when this segment is approached from below, one sees readily that $\overline{A_2^+} = A_2^-$. This allows us to reduce the problem to the solution of the following homogeneous Hilbert problem:

$$A_2^+ = GA_2^-, \qquad G = -\frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2} \frac{R(\theta)}{\overline{R(\theta)}} \tag{4.13}$$

whose general solution has the form

$$A_2(\theta) = i\beta \frac{X_0(\theta)}{\sqrt{a+\theta}\sqrt{d+\theta}}, \quad X_0(\theta) = \exp \int_{-d}^{-a} \frac{\ln G_1}{\xi - \theta} d\xi, \qquad G_1 = -G \tag{4.14}$$

with β a real constant. The radical is understood to denote the branches which are positive on the upper banks of the cuts $\theta > -a$ and $\theta > -d$. Analogously for $A_1(\theta)$ we obtain

$$A_1(\theta) = i\beta \frac{\theta + c_0}{\theta - \theta_0} \frac{X_0(\theta)}{\sqrt{a+\theta}\sqrt{d+\theta}} \tag{4.15}$$

The constants may be evaluated from a consideration of the singularities of the functions $\Phi_1'(\theta) + \Phi_2'(\theta)$ and $\Psi_1'(\theta) + \Psi_2'(\theta)$ at $\theta = \theta_0$. According to (4.6), letting θ tend to θ_0, we obtain

$$\beta = \frac{2(\lambda_1^\circ S_1^\circ + \lambda_2^\circ S_2^\circ)}{\pi(\theta_0 + c_0)} \sqrt{a + \theta_0} \exp\left(-\frac{1}{2\pi i} \int_{-d}^{-a} \frac{\ln G_1}{\xi - \theta_0} d\xi\right) \tag{4.16}$$

and the problem is entirely solved. Setting $c^{-2} = a^{-2} - d^{-2}$, we are led to the solution of the same problem for an isotropic body:

$$G_1 = -\frac{F(\theta)}{\overline{F(\theta)}}, \qquad F(\theta) = (d^2 - 2\theta^2)^2 + 4\theta^2 \sqrt{a^2 - \theta^2} \sqrt{d^2 - \theta^2}$$

$$\Phi_2'(\theta) = \Psi_1'(\theta) = 0 \tag{4.17}$$

$$\Phi_1'(\theta) = (a^{-2} - d^{-2}) \frac{d^2 - 2\theta^2}{F(\theta)} A(\theta), \qquad \Psi_2'(\Psi) = (a^{-2} - d^{-2}) \frac{\sqrt{a^2 - \theta^2}}{F(\theta)} A(\theta)$$

5. Diffraction by a rigid slit. For an isotropic body this problem has already been studied in [5] and [6]. The solutions of Equations (1.1), of the form (1.4), will be constructed for the displacements. According to [1], we obtain

$$\begin{aligned} u(x, y, t) &= \mathrm{Re}\,[u_1(\theta_1) + u_2(\theta_2)] = \sum_{j=1}^{2} \mathrm{Re} \int^{\theta_j} K(\xi)\, \lambda_j(\xi)\, \omega_j(\xi)\, d\xi \\ v(x, y, t) &= \mathrm{Re}\,[v_1(\theta_1) + v_2(\theta_2)] = \sum_{j=1}^{2} \mathrm{Re} \int^{\theta_j} L_j(\xi)\, \omega_j(\xi)\, d\xi \end{aligned} \tag{5.1}$$

where

$$L_j(\xi) = a^{-2}\xi^2 + d\lambda_j^2(\xi) - 1, \qquad K(\xi) = c^{-2}\xi \tag{5.2}$$

and the variables θ_j are defined by the relations (1.15). The elastic medium occupies the plane with a cut along $y = 0$, $x > 0$. For $t < 0$, in the left half-plane $x < 0$, we have a plane wave

$$u^\circ(x, y, t) = -K(\theta_0)\,[\lambda_1^\circ\omega_1^\circ(\Omega_1^\circ) + \lambda_2^\circ\omega_1^\circ(\Omega_2^\circ)] \tag{5.3}$$

$$v^\circ(x, y, t) = L_1(\theta_0)\,\omega_1^\circ(\Omega_1^\circ) + L_2(\theta_0)\,\omega_2^\circ(\Omega_2^\circ)$$

$$\Omega_j^\circ = t - \theta_0 x - \lambda_j^\circ y \qquad (0 < \theta_0 < a)$$

which impinges at the time $t = 0$ on the edge of the slit. The diffraction pattern for $t > 0$ is depicted in Fig. 3.

The reflection of the plane waves in the neighborhood of the lower

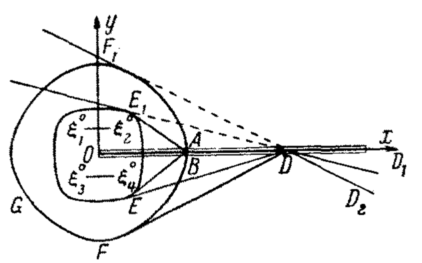

Fig. 3.

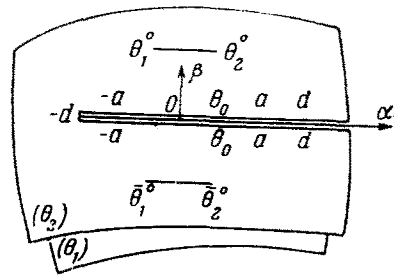

Fig. 4.

boundary of the slit may be obtained by a calculation of the boundary conditions corresponding to a wave packet of plane waves of the form

$$u^{\circ\circ}(x, y, t) = \sum_{j=1}^{2} K^\circ[-\lambda_j^\circ\omega_j^\circ(\Omega_j^\circ) + \lambda_j^\circ N_j\omega_j^\circ(\Omega_j^{\circ\prime}) + \lambda_{3-j}^\circ E_j\omega^\circ(\Omega_{3-j}^{\circ\prime})]$$

$$v^{\circ\circ}(x, y, t) = \sum_{j=1}^{2} [L_j^\circ\,\omega_j^\circ(\Omega_j^\circ) + L_j^\circ N_j\omega_j(\Omega_j^{\circ\prime}) + L_j^\circ E_j\omega_j^\circ(\Omega_{3-j}^{\circ\prime})]$$

$$\Omega_j^{\circ\prime} = t - \theta_0 x + \lambda_j^\circ y, \qquad K^\circ = K(\theta_0), \qquad L_j^\circ = L_j^\circ(\theta_0) \tag{5.4}$$

In order to fulfill the conditions for $y = 0$ we must have

$$N_j = \frac{\lambda_j^\circ L_{3-j}^\circ + \lambda_{3-j}L_j^\circ}{\lambda_j^\circ L_{3-j} - \lambda_{3-j}L_j}, \qquad E_j = \frac{2\lambda_j^\circ L_j^\circ}{\lambda_j^\circ L_{3-j}^\circ - \lambda_{3-j}^\circ L_j} \tag{5.5}$$

Let us formulate our boundary-value problem for the functions

$$u(\theta) = u_1(\theta) + u_2(\theta), \quad v(\theta) = v_1(\theta) + v_2(\theta) \quad \left(\theta = \frac{t}{x}\right) \tag{5.6}$$

i.e. let us find the values of these functions on the real axis, where the variables θ_1 and θ_2, defined by the relations (1.15), coincide. The functions u_j and v_j represent the disturbance in the domain AF_1GFB of the planes θ_j of the Riemann surface, where cuts have to be made, respectively, along the segments of the real axis $\theta_1 > -a$ and $\theta_2 > -d$ (see Fig. 4). Since the function $\omega_j^\circ(\xi)$ is a step function which equals zero for $\xi < 0$ and equals unity for $\xi > 0$, it follows that the functions Re $u(\theta)$ and Re $v(\theta)$ are piecewise constant on the boundaries of the cut, and that

$$\begin{gathered} \operatorname{Re} u(\theta) = \operatorname{Re} v(\theta) = 0 \qquad (\theta > \theta_0) \\ \operatorname{Re} u(\theta) = \alpha^\circ, \quad \operatorname{Re} v(\theta) = \beta^\circ \qquad (-a < \theta < \theta_0) \end{gathered} \tag{5.7}$$

where θ_0 on the upper bank of the cut corresponds to the points E and F, and on the lower bank corresponds to the points E_1 and F_1 (see Fig. 3):

$$\alpha^\circ = u_1^\circ + u_2^\circ = k^\circ(\lambda_1^\circ + \lambda_2^\circ), \qquad \beta^\circ = v_1^\circ + v_2^\circ = L_1^\circ + L_2^\circ \tag{5.8}$$

Performing a cut in the plane θ_1 along the segment $(-d, -a)$, and denoting by f_1 and f_2 respectively the real values of $u(\theta)$ and $v(\theta)$ along this segment, we obtain readily, as in [5]

$$\begin{aligned} u'(\theta) &= \frac{\alpha^\circ}{\pi i}\frac{1}{\theta-\theta_0}\frac{\sqrt{d+\theta_0}}{\sqrt{d+\theta}} + \frac{1}{\pi i\sqrt{d+\theta}}\int_{-d}^{-a}\frac{\sqrt{d+\xi}f_1'}{\xi-\theta}\,d\xi \\ v'(\theta) &= \frac{\beta^\circ}{\pi i}\frac{1}{\theta-\theta_0}\frac{\sqrt{d+\theta_0}}{\sqrt{d+\theta}} + \frac{1}{\pi i\sqrt{d+\theta}}\int_{-d}^{-a}\frac{\sqrt{d+\xi}}{\xi-\theta}f'_2\,d\xi \end{aligned} \tag{5.9}$$

Let us put (5.10)

$$A(\theta) = \frac{\alpha^\circ}{\pi i}\sqrt{d+\theta_0} + (\theta-\theta_0)\Phi^{\circ\circ}(\theta), \qquad \Phi^{\circ\circ}(\theta) = \frac{1}{\pi i}\int_{-d}^{-a}\frac{\sqrt{d+\xi}}{\xi-\theta}f_1'\,d\xi$$

$$B(\theta) = \frac{\beta_0}{\pi i}\sqrt{d+\theta_0} + (\theta-\theta_0)\Psi^{\circ\circ}(\theta), \qquad \Psi^{\circ\circ}(\Psi) = \frac{1}{\pi i}\int_{-d}^{-a}\frac{\sqrt{d+\xi}f_2'}{\xi-\theta}\,d\xi$$

According to (5.1) we have

$$K(\theta)\,[\lambda_1\omega_1 + \lambda_2\omega_2] = \frac{A(\theta)}{(\theta-\theta_0)\sqrt{d+\theta}}, \qquad L_1\omega_1 + L_2\omega_2 = \frac{B(\theta)}{(\theta-\theta_0)\sqrt{d+\theta}} \tag{5.11}$$

which implies

$$\omega_1(\theta) = \frac{L_2 A(\theta) - K\lambda_2 B(\theta)}{(\theta - \theta_0)\sqrt{d+\theta}\,\Delta_1 K} \tag{5.12}$$

$$K\Delta_1(\theta) = \lambda_1 L_2 - \lambda_2 L_1 = \frac{\theta\sqrt{a^2-\theta^2}}{a^2 b^2 c^2}(\lambda_2 - \lambda_1)\,[d^2\sqrt{a^2-\theta^2} + a^2\sqrt{d^2-\theta^2}]$$

Since the function ω_1 may be continued analytically across the segment $(-d, -a)$, i.e. its limiting values from above and below this segment must coincide: $\omega_1^+ = \omega_1^-$, we are led to the equation

$$L_2\left(\frac{A^+}{\Delta_1^+} + \frac{A^-}{\Delta_1^-}\right) = k\lambda_2\left(\frac{B^+}{\Delta_1^+} - \frac{B^-}{\Delta_1^-}\right) \tag{5.13}$$

The right-hand side of this equation is real, while the left-hand side is purely imaginary, because $A^- = -\overline{A^+}$, $B^- = \overline{-B^+}$, $\Delta_1^- = -\overline{\Delta_1^+}$, and L_2 and K are real on the segment in question; consequently, this equation is equivalent to the following two equations:

$$\begin{matrix} A^+ = G'A^- \\ B^+ = G_1'B^- \end{matrix} \qquad \left(C' = -G_1' = \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1}\,\frac{d^2\sqrt{a^2-\theta^2} + a^2\sqrt{d^2-\theta^2}}{d^2\sqrt{a^2-\theta^2} - a^2\sqrt{d^2-\eta^2}}\right) \tag{5.14}$$

The solution of these equations, which is bounded at infinity and in the neighborhood of the boundary points, has the form

$$A(\theta) = i\alpha^{\circ\circ}\frac{\sqrt{d+\theta}}{\sqrt{a+\theta}}Y_0(\theta), \quad B(\theta) = i\beta^{\circ\circ}Y_0(\theta), \quad Y_0(\theta) = \exp\frac{1}{2\pi i}\int_{-d}^{-a}\frac{\ln G_1'}{\xi - \theta}\,d\xi \tag{5.15}$$

The function $\omega_1(\theta)$, and together with it the functions $u_1'(\theta)$ and $v_1'(\theta)$, is holomorphic in the neighborhood of the point $\theta = -d$, and satisfies

$$|\omega_1(\theta)| < \frac{N}{|\theta + d|^\gamma}$$

where N and γ are real constants, with $\gamma < 1$. Consequently, the point $\theta = -d$ is a removable singularity for this function. The constants $\alpha^{\circ\circ}$ and $\beta^{\circ\circ}$ may be obtained by comparing (5.10) and (5.15) for $\theta = \theta_0$; the result is

$$\alpha^{\circ\circ} = -\alpha^\circ\frac{\sqrt{a+\theta_0}}{\pi Y_0(\theta_0)}, \qquad \beta^{\circ\circ} = -\beta^\circ\frac{\sqrt{d+\theta_0}}{\pi Y_0(\theta_0)}$$

If, instead, we choose the functions $\Phi^{\circ\circ}(\theta)$ and $\Psi^{\circ\circ}(\theta)$ as unknown functions, we obtain nonhomogeneous equations of the type studied in [5]. However, in this case the structure of the solution is much more complicated than in (5.13) above, where the whole matter reduces to the calculation of a single function $Y_0(\theta)$, which may be given in the form of tables [6].

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